Pergamon

J. Appl. Maths Mechs, Vol. 62, No. 2, pp. 177–186, 1998
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0021–8928/98/$—see front matter

PII: S0021–8928(98)00023-9

THE PROBLEM OF COMPUTING THE VALUE OF A DIFFERENTIAL GAME FOR A POSITIONAL FUNCTIONAL†

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(*Received* 28 *August* 1997)

A dynamical system driven by controls and uncontrollable noise is considered in a game-theoretic setting [1–8]. The problem of feedback control in which the performance index is a positional functional of the motion of the system [8–11] is investigated. On the assumption that the structure of the functional satisfies reasonably general conditions, a procedure is proposed for computing the value of the corresponding differential game. Irrespective of the number of dimensions in the initial problem, as dictated by the structure of the performance index, the proposed procedure reduces to the problem of the successive construction of the upper convex hulls of certain auxiliary functions in domains whose dimension does not exceed that of the phase vector of the system. © 1998 Elsevier Science Ltd. All rights reserved.

Consider a two-person differential game for a conflict-control system described by ordinary differential equations that are linear in the phase vector x and let the performance index be a semi-norm in the function space of the motions $x[\cdot]$. It has been shown [11] that the calculation of the value of the game can be reduced to successive construction of the upper convex hulls $\varphi_j(\cdot)$ of certain auxiliary functions $\psi_j(x)$ defined in suitable domains G_j ($j = k, k-1, \ldots, 1$, where k is a fairly large natural number) in a suitable space of dual variables. The efficacy of the procedure depends essentially on the dimension and structure of the latter space, namely, on the dimension of the variables with respect to which the convexification is carried out.

Generally speaking, the appropriate space is built up from the space of vectors m dual to the phase vector x and a space of additional parameters dual to specific finite-dimensional information elements of the prehistory of the motion. The number and dimension of the additional parameters depend on the specific properties of the performance index. For example, in a game in which the performance index is the total deviation of the motion $x[\cdot]$ from a given trajectory (see [8, p. 86], and also [10]), no additional parameters are needed and the constructions take place in the space of the vectors m. In a game with a performance index such as the maximum deviation of $x[\cdot]$ from a given trajectory [8, p. 92], the constructions involve an additional scalar parameter ν. However, it is sufficient to construct the hulls $\varphi_j(m, \nu)$ for the functions $\psi_j(m, \nu)$ by convexification of the latter with respect to m only in domains $G_{j,\nu}$, for fixed values of ν. Thus, here too the constructions are actually carried out in the space of the vectors m.

Characteristically, the performance indices used are positional functionals [8, p. 43; 9]. Hence the role of the information image for the optimal strategies in these games is played by the actual state $\{t, x[t]\}$ of the system. On the other hand, in a game in which the performance index is the sum of the maximum and the total deviations of the motion from a given trajectory [11, p. 891], which is not a positional functional, computation of the value of the game requires the construction of upper convex hulls $\varphi_j(m, \nu)$ for suitable functions $\psi_j(m, \nu)$ in domains G_j of pairs $\{m, \nu\}$. Here convexification of the functions $\psi_j(m, \nu)$ with respect to m only is no longer sufficient (there is a counterexample). Note that in such a game the whole prehistory of the motion plays the part of the information image for the strategies that form the saddle point of the game.

The discussion then concentrates in detail on the case when the performance index considered in [11] has what is known as positional structure (and, as a corollary, it is a positional functional). It is proved that in this fairly general case, as in the special cases mentioned previously [8, pp. 86, 92], computation of the value of the game reduces to constructing upper hulls of functions defined in domains of a space consisting only of dual vectors m (i.e. convexification with respect to the additional parameters is not necessary).

†*Prikl. Mat. Mekh.* Vol. 62, No. 2, pp. 188–198, 1998.

1. STATEMENT OF THE PROBLEM. CONSIDER THE SYSTEM

$$dx/dt = A(t)x + f(t,u,v), \quad t_0 \leqslant t \leqslant \vartheta \tag{1.1}$$

$$x \in R^n, \quad u \in P \subset R^r, \quad v \in Q \subset R^s$$

Here x is the phase vector, u is the control vector, v is the noise vector, t_0 and ϑ are given times ($t_0 < \vartheta$), P and Q are known compact sets, and $A(t)$ and $f(t, u, \upsilon)$ are a matrix-valued function and a vector-valued function, respectively, both piecewise continuous with respect to t.

Throughout this paper a function $F(t, z)$, $t \in [t_0, \vartheta]$, $z \in Z$ is said to be piecewise continuous with respect to t if there are a finite number of points t_q of discontinuity with respect to t, which do not depend on z, and moreover in the intervals where $F(t, z)$ is continuous in t it is jointly continuous in all arguments, while at the points of discontinuity t_q it is continuous from the right and may be defined from the left in such a way that the resulting function is jointly continuous in $[t_{q-1}, t_q] \times Z$.

The saddle-point condition for a small game [1, 2; 3, p. 56] is satisfied, i.e. for any $m \in R^n$ and $t \in [t_0, \vartheta]$ it is true that

$$\min_{u\in P}\max_{v\in Q}\langle m, f(t,u,v)\rangle = \max_{v\in Q}\min_{u\in P}\langle m, f(t,u,v)\rangle \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product of vectors.

We will call a position of system (1.1) a pair $[t, x]$. Suppose given some position $\{t_*, x_*\}$, $t_0 \leqslant t_* < \vartheta$. Borel-measurable realizations $u[t_*[\cdot]\vartheta) = \{u[t] \in P, t_* \leqslant t < \vartheta\}$ and $\upsilon[t_*[\cdot]\vartheta) = \{\upsilon[t] \in Q, t_* \leqslant t < \vartheta\}$ are admissible. Beginning from a position $[t_*, x_*]$, such realizations generate, according to (1.1) (with $u = u[t]$, $\upsilon = \upsilon[t]$), absolutely continuous motions $x[t_*[\cdot]\vartheta] = \{x[t], t_* \leqslant t \leqslant \vartheta, x[t_*] = x_*\}$. We shall assume that, in the space of the variables t, x, we are given a compact set K (see, for example, [7, p. 40]) of possible positions of system (1.1). The projection of K on to the t axis is an interval $[t_0, \vartheta]$. It is also assumed that all trajectories of system (1.1) that begin at an arbitrary position $\{t_*, x_*\} \in K$ remain in K for all $t \in (t_*, \vartheta]$. Then the motions under consideration will satisfy a Lipschitz condition with respect to t with the same constant λ_K.

The performance index γ of the motion $x[t_*[\cdot]\vartheta]$ is defined as a functional $\gamma(x[t_*[\cdot]\vartheta])$ which has the following structure.

Let the following be given: a natural number $N \geqslant 1$, times $t^{[i]} \in [t_0, \vartheta]$, $t^{[i]} < t^{[i+1]}$ ($i = 1, \ldots, N-1$), $t^{[N]} = \vartheta$, constant ($d^{[i]} \times n$) matrices $D^{[i]}$ ($1 \leqslant d^{[i]} \leqslant n$) and n-dimensional vectors $g^{[i]}$ ($i = 1, \ldots, N$). It is assumed that the rows of the matrices $D^{[i]}$ are linearly independent. We consider the spaces of ($d^{[i]} + \ldots + d^{[N]}$)-dimensional vector-tuples $\{y^{[i]}, \ldots, y^{[N]}\}$ consisting of $d^{[q]}$-dimensional vectors $y^{[q]}$ ($q = i, \ldots, N$), assuming certain norms $\mu^{[i]}(\{y^{[i]}, \ldots, y^{[N]}\})$ ($i = 1, \ldots, N$; when $i = N$ the symbol $\{y^{[i]}, \ldots, y^{[N]}\}$ simply stands for the vector $y^{[N]}$) defined in these spaces.

The functional γ is then has the form

$$\gamma = \gamma(x[t_*[\cdot]\vartheta]) = \mu^{[h(t_*)]}(\{D^{[h(t_*)]}(x[t^{[h(t_*)]}] - g^{[h(t_*)]}), \ldots, D^{[N]}(x[t^{[N]}] - g^{[N]})\}) \tag{1.3}$$

Here

$$h(t_*) = \min_{i=1,\ldots,N}\{i : t^{[i]} \geqslant t_*\} \tag{1.4}$$

We shall assume in addition that functions $\sigma^{[i]}(y^{[i]}, \beta)$ $y^{[i]} \in R^{d^{[i]}}$, $\beta \in R$, $\beta \geqslant 0$, exist such that

$$\mu^{[i]}(\{y^{[i]}, \ldots, y^{[N]}\}) = \sigma^{[i]}(y^{[i]}, \beta'), \quad \beta' = \mu^{[i+1]}(\{y^{[i+1]}, \ldots, y^{[N]}\})$$
$$i = 1, \ldots, N-1 \tag{1.5}$$

We may assume without loss of generality that the functions $\sigma^{[i]}(y^{[i]}, \beta)$ are defined for $\beta < 0$ in such a way that they are even functions of β, that is, for $\beta < 0$ we put $\sigma^{[i]}(y^{[i]}, \beta) = \sigma^{[i]}(y^{[i]}, -\beta)$, $i = 1, \ldots, N-1$. It then follows from (1.5) that the functions $\sigma^{[i]}(y^{[i]}, \beta)$ are norms in the spaces of ($d^{[i]} + 1$)-dimensional vectors $\{y^{[i]}, \beta\}$.

Now, whatever the history $x[t_*[\cdot]t^*) = \{x[\tau], t_* \leqslant \tau < t^*\}$, $t^* \leqslant \vartheta$ of a motion $x[t_*[\cdot]\vartheta]$ of system (1.1), the functional γ of (1.3) may be expressed as

$$\gamma(x[t_*[\cdot]\vartheta]) = \sigma(x[t_*[\cdot]t^*), \beta^*), \quad \beta^* = \gamma(x[t^*[\cdot]\vartheta])$$

Here, if $h(t_*) = h(t^*)$, we have $\sigma(x[t_*[\cdot]t^*), \beta^*) = \beta^*$, while if $h(t_*) < h(t^*)$ we put $\sigma(x[t_*[\cdot]t^*), \beta^*) = \sigma_{[h_*]}^{[h^*]}(x[t_*[\cdot]t^*), \beta^*)$, $h_* = h(t_*)$, $h^* = h(t^*)$, where $\sigma_{[h_*]}^{[h^*]}(x[t_*[\cdot]t^*), \beta^*)$ are defined from the recurrent relations

$$\sigma_{[h^*-1]}^{[h^*]}(x[t_*[\cdot]t^*), \beta^*) = \sigma^{[h^*-1]}(D^{[h^*-1]}(x[t^{[h^*-1]}] - g^{[h^*-1]}), \beta^*)$$

$$\sigma_{[i]}^{[h^*]}(x[t_*[\cdot]t^*), \beta^*) = \sigma^{[i]}(D^{[i]}(x[t^{[i]}] - g^{[i]}), \sigma_{[i+1]}^{[h^*]}(x[t_*[\cdot]t^*), \beta^*))$$

$$i = h_*, \ldots, h^* - 2$$

It follows from the properties of the functions $\sigma^{[i]}(y^{[i]}, \beta)$ enumerated above that, for any fixed history $x[t_*[\cdot]t^*)$, the functional $\sigma(x[t_*[\cdot]t^*), \beta^*)$ is a non-decreasing function of β_* ($\beta_* \geqslant 0$). Thus, when condition (1.5) is satisfied, the functional (1.3) is positional [8, p. 43; 9].

It is required to find a control (or noise) designed to minimize (maximize) the index γ (1.3), (1.5).

These two problems combine, as shown in [7, p. 75; 8, p. 51], into a two-person zero-sum differential game (u is the move of Player I and υ is that of Player II) in the class of purely positional universal strategies $u(t, x, \varepsilon)$ and $\upsilon(t, x, \varepsilon)$, where $\{t, x\} \in K$ and $\varepsilon > 0$ is a precision parameter. It follows from condition (1.2) and from the fact that γ is a positional functional that, whatever the initial position $[t_*, x_*\} \in K$, the game just defined has a value $\rho(t_*, x_*)$. Moreover, the game has a saddle point, made up of an optimal minimax strategy $u^\circ(t, x, \varepsilon)$ and an optimal maximin strategy $\upsilon^\circ(t, x, \varepsilon)$. By the definitions of the value of a game and of optimal strategies, it follows that for any number $\zeta > 0$ a number $\varepsilon(\zeta) > 0$ and a function $\delta(\zeta, \varepsilon) > 0$, $0 < \varepsilon \leqslant \varepsilon(\zeta)$ exist such that, whatever the initial position $[t_*, x_*\} \in K$, $t_* < \vartheta$, the number $\varepsilon > 0$, $\varepsilon \leqslant \varepsilon(\zeta)$ and the partition $\Delta_M\{t_j\} = \{t_j : t_1 = t_*, t_j < t_{j+1}, j = 1, \ldots, M, t_{M+1} = \vartheta\}$ of the mesh $\delta_M = \max_{j=1,\ldots,M}(t_{j+1} - t_j) \leqslant \delta(\zeta, \varepsilon)$ on the one hand, the stepwise control law $U^\circ = \{u^\circ(\cdot), \varepsilon, \Delta_M\{t_j\}\}$ producing the control impulse

$$u^\circ[t] = u^\circ(t_j, x[t_j], \varepsilon), \quad t_j \leqslant t < t_{j+1}, \quad j = 1, \ldots, M \tag{1.6}$$

guarantees the inequality

$$\gamma \leqslant \rho(t_*, x_*) + \zeta$$

whatever the admissible realization $\upsilon[t_*[\cdot]\vartheta)$ of the noise happens to be; on the other hand, the stepwise law $V^\circ = \{\upsilon^\circ(\cdot), \varepsilon, \Delta_M\{t_j\}\}$ producing the noise impulse

$$\upsilon^\circ[t] = \upsilon^\circ(t_j, x[t_j], \varepsilon), \quad t_j \leqslant t < t_{j+1}, \quad j = 1, \ldots, M \tag{1.7}$$

guarantees the inequality

$$\gamma \geqslant \rho(t_*, x_*) - \zeta$$

whatever the admissible realization $u[t_*[\cdot]\vartheta)$ of the control happens to be.

Optimal strategies $u^\circ(t, x, \varepsilon)$ and $\upsilon^\circ(t, x, \varepsilon)$ are constructed as extremals (see [7, pp. 210, 220] or [8, pp. 62–64] to the value function $\rho(t, x)$. Thus, to produce an optimal control and a counter-optimal noise, it is sufficient to be able to compute effectively the value of the game with any position $[t, x]$ acting as initial position. This is the aim of the present paper.

Remark 1. If condition (1.2) does not hold in the differential game under consideration, the solution process is transferred to the class of mixed strategies [8, p. 247; 9]. However, the auxiliary constructions proposed in this paper also remain one of the main elements in those more complicated constructions.

Remark 2. The performance index (1.3) may be given from the start, or introduced as an approximation for the initial index $\gamma_*(x[t_*[\cdot]\vartheta])$, which takes a continuum of values of $x[t]$ into account. For example, let the performance index be

$$\gamma_*^{(p)} = \gamma_*^{(p)}(x[t_*[\cdot]\vartheta]) = \left(\int_{t_*}^{\vartheta} [\chi(t, D(t)(x[t] - g(t)))]^p \, dt \right)^{1/p} \tag{1.8}$$

where p is a given number ($1 < p < \infty$), $g(t)$ is a known piecewise continuous n-dimensional vector-valued function, $D(t)$ is a given piecewise constant ($d(t) \times n$) matrix-valued function ($1 \leqslant d(t) \leqslant n$) and $\chi(t, D(t)x)$ is a semi-norm, which is a piecewise continuous function of $t(\chi(t, \cdot)$ is a norm in the space of $d(t)$-dimensional vectors for each fixed t).

The functional (1.8) is positional. Let $\rho_*^{(p)}(t_*, x_*)$ be the value and $u^\circ_{(p)*}(t, x, \varepsilon)$ and $\upsilon^\circ_{(p)*}(t, x, \varepsilon)$ optimal strategies in a differential game for system (1.1), (1.2) with the performance index $\gamma_*^{(p)}$.

A functional $\gamma^{(p)}$ approximating $\gamma_*^{(p)}$ may be constructed as follows. Consider some given partition

$$\Delta_N\{t^{[i]}\} = \{t^{[i]} : t^{[0]} = t_0,\ t^{[i-1]} < t^{[i]},\ i = 1, \ldots, N,\ t^{[N]} = \vartheta\} \tag{1.9}$$

of the interval $[t_0, \vartheta]$ which includes all the times defining the intervals over which the matrix-valued function $D(t)$ is constant, as well as all points at which the functions $g(t)$ and $\chi(t, D(t)x)$ in (1.8) are discontinuous with respect to t. Put

$$\begin{gathered} D^{[i]} = D(t^{[i]})(t^{[i]} - t^{[i-1]})^{1/p}, \quad d^{[i]} = d(t^{[i]}), \quad g^{[i]} = g(t^{[i]}) \\ \chi^{[i]}(y^{[i]}) = \chi(t^{[i]}, y^{[i]}), \quad y^{[i]} \in R^{d^{[i]}}, \quad i = 1, \ldots, N \end{gathered} \tag{1.10}$$

We further assume

$$\gamma^{(p)} = \gamma^{(p)}(x[t_*[\cdot]\vartheta]; \Delta_N\{t^{[i]}\}) = \left(\sum_{i=h(t_*)}^{N} \left[\chi^{[i]}(D^{[i]}(x[t^{[i]}] - g^{[i]}))\right]^p \right)^{1/p} \tag{1.11}$$

where $h(t_*)$ is defined by (1.4).

Then for any $\zeta > 0$ a $\delta(\zeta) > 0$ exists such that, for any partition $\Delta_N\{t^{[i]}\}$ (1.9) of mesh $\delta_N = \max_{i=1,\ldots,N}(t^{[i]} - t^{[i-1]}) \leqslant \delta(\zeta)$

$$\left|\gamma_*^{(p)}(x[t_*[\cdot]\vartheta]) - \gamma^{(p)}(x[t_*[\cdot]\vartheta];\ \Delta_N\{t^{[i]}\})\right| \leqslant \zeta \tag{1.12}$$

whatever the vector-valued function $x[t_*[\cdot]\vartheta] = \{x[t], \{t, x[t]\} \in K, t_* \leqslant t \leqslant \vartheta\}, t_* \in [t_0, \vartheta]$, provided it satisfies a Lipschitz condition with respect to t with constant λ_K.

The functional $\gamma^{(p)}$ has the structure (1.3), (1.5). In this case the times $t^{[i]}$ ($i = 1, \ldots, N$) are defined by the choice of the partition $\Delta_N\{t^{[i]}\}$ of (1.9), of the matrix $D^{[i]}$ and the vector $g^{[i]}$ of (1.10); while the norms $\mu^{[i]}(\cdot)$ and functions $\sigma^{[i]}(\cdot)$ are defined by

$$\mu^{[i]}(\{y^{[i]}, \ldots, y^{[N]}\}) = \left(\sum_{q=i}^{N} \left[\chi^{[q]}(y^{[q]})\right]^p \right)^{1/p}, \quad i = 1, \ldots, N$$

$$\sigma^{[i]}(y^{[i]}, \beta) = \left([\chi^{[i]}(y^{[i]})]^p + |\beta|^p \right)^{1/p}, \quad i = 1, \ldots, N-1$$

Let $\rho^{(p)}(t_*, x_*; \Delta_N\{t^{[i]}\})$ be the value and $u^\circ_{(p)}(t, x, \varepsilon; \Delta_N\{t^{[i]}\})$ and $\upsilon^\circ_{(p)}(t, x, \varepsilon; \Delta_N\{t^{[i]}\})$ the optimal strategies in a game for system (1.1), (1.2) with performance index $\gamma^{(p)}$ of (1.11) (for some sufficiently fine fixed partition $\Delta_N\{t^{[i]}\}$ (1.9)). Considering the motions of system (1.1), which are realized in the case when Player I is guided by strategy $u^\circ_{(p)*}(\cdot)$, and Player II by strategy $\upsilon^\circ_{(p)}(\cdot\,; \Delta_N\{t^{[i]}\})$, and conversely in the case when the players adopt strategies $u^\circ_{(p)}(\cdot\,; \Delta_N\{t^{[i]}\})$ and $\upsilon^\circ_{(p)}(\cdot)$, respectively, we conclude, by (1.12), that for any $\eta > 0$ a $\delta(\eta) > 0$ exists such that, for any partition $\Delta_N\{t^{[i]}\}$ as in (1.9) of mesh $\delta_N = \max_{i=1,\ldots,N}(t^{[i]} - t^{[i-1]}) \leqslant \delta(\eta)$

$$\left|\rho_*^{(p)}(t_*, x_*) - \rho^{(p)}(t_*, x_*; \Delta_N\{t^{[i]}\})\right| \leqslant \eta$$

whatever the position $\{t_*, x_*\} \in K$.

Thus, the problem of constructing a minimax (maximin) control for system (1.1), (1.2) with performance index $\gamma_*^{(p)}$ (1.8) reduces to constructing the value function $\rho^{(p)}(t, x; \Delta_N\{t^{[i]}\})$ of a differential game for a functional $\gamma^{(p)}$ (1.11) of structure (1.3), (1.5).

2. A PROCEDURE FOR COMPUTING THE VALUE OF THE GAME

Thus, let us consider a differential game for system (1.1), (1.2) with performance index (1.3), (1.5). Suppose that a position $\rho^{(p)}(t, x; \Delta_N\{t^{[i]}\})$ is realized. Let $t_* < \vartheta$. We assign a partition

$$\Delta_k = \Delta_k\{\tau_j\} = \{\tau_j : \tau_1 = t_*,\ \tau_j < \tau_{j+1},\ j = 1, \dots, k,\ \tau_{k+1} = \vartheta\} \tag{2.1}$$

of the time interval $[t_*, \vartheta]$ in which we include all points t at which the functions $A(t)$ and $f(t, u, \upsilon)$ are discontinuous, as well as all times $t^{[i]}$ $(i = h(t_*), \dots, N)$ of (1.3), (1.5). Let $X(t, \tau)$ be a fundamental solution matrix for the equation $dx/dt = A(t)x$. We put

$$\Delta\psi_j(t_*, m) = \int_{\tau_j}^{\tau_{j+1}} \max_{\upsilon\in Q} \min_{u\in P} \langle m, X(\vartheta,\tau) f(\tau, u, \upsilon)\rangle d\tau \tag{2.2}$$

$$m \in R^n, \quad j = 1, \dots, k$$

Moving in retrograde fashion via the division points of the partition $\Delta_k\{\tau\}$ (2.1), we will construct a sequence of domains $G_j(t_*, \tau_j \pm 0)$ in the space R^n of vectors m and a sequence of functions $\varphi_j(t_*, \tau_j \pm 0, m)$, $m \in G_j(t_*, \tau_j \pm 0)$ $(j = k+1, k, \dots, 1)$.

Specifically, for $j = k + 1$ we assume

$$\begin{aligned} &G_{k+1}(t_*, \tau_{k+1} + 0) = \{m : m = 0\}, \quad \varphi_{k+1}(t_*, \tau_{k+1} + 0, m) \equiv 0 \\ &G_{k+1}(t_*, \tau_{k+1} - 0) = \left\{m : m = D^{[N]T} l,\ l \in R^{p^{[N]}},\ \mu^{[N]*}(l) \leqslant 1\right\} \\ &\varphi_{k+1}(t_*, \tau_{k+1} - 0, m) = -\langle m, g^{[N]}\rangle, \quad m \in G_{k+1}(t_*, \tau_{k+1} - 0) \end{aligned} \tag{2.3}$$

The superscript T denotes transposition and $\mu^{[N]*}(\cdot)$ is the norm dual to the norm $\mu^{[N]*}(\cdot)$ of (1.3). Throughout, the following conventions will be used when describing domains: the notation for an element appears first in the braces, then a colon, and then the conditions for an element to belong to the domain (thus, the domain $G_{k+1}(t_*, \tau_{k+1} + 0)$ consists of the single element $m = 0$; while $m \in G(t_*, \tau_{j+1} - 0)$ if and only if m is such that a $p^{[N]}$-dimensional vector l exists, $\mu^{[N]*}(l) \leqslant 1$, for which $m = D^{[N]T} l$).

We continue by induction. Suppose that for $1 < j + 1 \leqslant k + 1$ the domains $G_{j+1}(t_*, \tau_{j+1} \pm 0)$ and the functions $\varphi_{j+1}(t_*, \tau_{j+1} \pm 0, m)$ have already been constructed. We then define for the next j value

$$\begin{aligned} &G_j(t_*, \tau_j + 0) = G_{j+1}(t_*, \tau_{j+1} - 0) \\ &\psi_j(t_*, m) = \Delta\psi_j(t_*, m) + \varphi_{j+1}(t_*, \tau_{j+1} - 0, m), \quad m \in G_j(t_*, \tau_j + 0) \\ &\varphi_j(t_*, \tau_j + 0, m) = \{\psi_j(t_*, \cdot)\}^*_{G_j(t_*, \tau_j + 0)} \end{aligned} \tag{2.4}$$

The symbol $\{\psi(\cdot)\}^*_G$ denotes the upper convex hull of the function $\psi(m)$, constructed by convexification with respect to m in G—that is, by definition, the minimum concave function of m that majorizes the function $\psi(m)$, $m \in G$.

Now, if τ_j is not one of the times $t^{[i]}$ of (1.3) (by (1.4), this means that $\tau_j < t^{[h]}$, where $h = h(\tau_j)$), we define

$$G_j(t_*, \tau_j - 0) = G_j(t_*, \tau_j + 0), \quad \varphi_j(t_*, \tau_j - 0, m) \equiv \varphi_j(t_*, \tau_j + 0, m) \tag{2.5}$$

But if $\tau_j = t^{[h]}$, $h = h(\tau_j)$, we define

$$G_j(t_*,\tau_j-0)=\{m:\ m=\nu m_*+X^T(t^{[h]},\vartheta)D^{[h]T}l,$$

$$\nu\geq 0,\ l\in R^{p^{[h]}},\ \sigma^{[h]*}(l,\nu)\leq 1,\ m_*\in G_j(t_*,\tau_j+0)\} \tag{2.6}$$

where $\sigma^{[h]*}(\cdot)$ is the norm dual to the norm $\sigma^{[h]*}(\cdot)$ of (1.5) (for $i = h$);

$$\varphi_j(t_*,\tau_j-0,m)=\max_{\{\nu,m_*,l\}|m}[\nu\varphi_j(t_*,\tau_j+0,m_*)-\langle l,D^{[h]}g^{[h]}\rangle] \tag{2.7}$$

$$m\in G_j(t_*,\tau_j-0)$$

where the maximum is taken over all possible triples $\{\nu, m_*, l\}$ associated by (2.6) with the given vector $m \in G_j(t_*, \tau_j - 0)$.

To complete the induction process, we deal with the case $j = 1$, constructing domains $G_1(t_*, \tau_1 \pm 0)$ and functions $\varphi_1(t_*, \tau_1 \pm 0, m)$, $m \in G_1(t_*, \tau_1 \pm 0)$.

It can be verified that for any j ($j = k + 1, k, \dots, 1$) the domains $G_j(t_*, \tau_j \pm 0)$ thus constructed are convex compact subsets of R^n containing the vector $m = 0$, while the functions $\varphi_j(t_*, \tau_j \pm 0, m)$ are concave, bounded and upper semi-continuous in their domains of definition, where

$$\varphi_j(t_*,\tau_j\pm 0,0)\geq 0 \tag{2.8}$$

We introduce the quantities

$$e(t_*\pm 0,x_*;\Delta_k)=\max_{m\in G_1(t_*,\tau_1\pm 0)}\alpha_1^{\pm}(t_*,x_*,m) \tag{2.9}$$

$$\alpha_q^{\pm}(t_*,w,m)=\langle m,X(\vartheta,t_*)w\rangle+\varphi_q(t_*,\tau_q\pm 0,m),\quad q=1,2$$

If $t_* = \vartheta$, we let Δ_k in (2.9) denote a "degenerate" partition consisting of the single point $\tau_1 = t_* = \vartheta = \tau_{k+1}$, and we then have domains $G_1(t_*, \tau_1 \pm 0) = G_{k+1}(t_*, \tau_{k+1} \pm 0)$ (and functions $\varphi_1(t_*, \tau_1 \pm 0, m) \equiv \varphi_{k+1}(t_*, \tau_{k+1} \pm 0, m)$ (see (2.3)). Then

$$e(\vartheta-0,x_*;\Delta_k)=\mu^{[N]}(D^{[N]}(x_*-g^{[N]}))=\gamma(x[\vartheta[\vartheta]\vartheta])$$

Theorem. For any number $\xi > 0$ a number $\delta(\xi) > 0$ exists such that, for any initial position $\{t_*, x_*\} \in K$ and partition Δ_k of the time interval $[t_*, \vartheta]$ with mesh $\delta_k = \max_{j=1,\dots,k}(\tau_{j+1} - \tau_j) \leq \delta(\xi)$, the following inequality holds

$$|\rho(t_*,x_*)-e(t_*-0,\ x_*;\Delta_k)|\leq\xi$$

where $\rho(t_*, x_*)$ is the value of the differential game for system (1.1), (1.2) with the performance index γ of (1.3), (1.5), and $e(t_*- 0, x_*; \Delta_k)$ is as defined in (2.9).

Thus, the procedure described above for computing the quantity $e(t_*-0, x_*; \Delta_k)$ on the basis of the functions $\varphi_j(t_*, \tau_j \pm 0, m)$, $m \in G_j(t_*, \tau_j \pm 0)$ ($j = 1, \dots, k + 1$) leads to the value of the differential game. Just as in the special cases considered in [8, pp. 117, 129] and [10], the control impulses $u^\circ(t_j, x[t_j], \varepsilon)$ in (1.6) and $\upsilon^\circ(t_j, x[t_j], \varepsilon)$ in (1.7) may be constructed effectively as extremals to the quantity $e(t_{j*}-0, \cdot)$ of (2.9). We emphasize that during the constructions, irrespective of the number N of instants of time $t^{[i]}$ in (1.3), the auxiliary functions $\psi_j(t_*, m)$ are defined and convexified in domains $G_j(t_*, \tau_j + 0)$ whose dimensions do not exceed that of the phase vector of the original system.

3. VALIDATION OF THE RESULT

The following two lemmas establish the necessary properties of the quantities $e(t_* \pm 0, x_*; \Delta_k)$.

Lemma 1. For any position $\{t_*, x_*\} \in K$, $t_* < \vartheta$ and partition Δ_k (2.1) of the time interval $[t_*, \vartheta]$:

$$e(t_* - 0, x_*; \Delta_k) = \begin{cases} e(t_* + 0, x_*; \Delta_k), & t_* < t^{[h]} \\ \sigma^{[h]}(D^{[h]}(x_* - g^{[h]}),\ e(t_* + 0, x_*; \Delta_k)), & t_* = t^{[h]} \end{cases}$$

where $h = h(t_*)$ as in (1.4).

Proof. In the case $t_* < t^{[h]}$ the assertion of the lemma follows from (2.5) (in the case $j = 1$) and (2.9), provided one takes into account that $\tau_1 = t_*$.

Let $t_* = t^{[h]}$. Then the domains $G_1(t_*, \tau_1 \pm 0)$ are related by (2.6) and the functions $\varphi_1(t_*, \tau_1 \pm 0, m)$ by the equality (2.7).

By (2.9), a vector $m_*^0 \in G_1(t_*, \tau_1 + 0)$ exists such that

$$e(t_* + 0, x_*; \Delta_k) = \alpha_1^+(t_*, x_*, m_*^0) \tag{3.1}$$

We recall that $\sigma^{[h]}(y^{[h]}, \beta)$ is an even function of β, and hence $\sigma^{[h]*}(l, \nu)$ is an even function of ν. In addition, by (2.8), $e(t_* + 0, x_*; \Delta_k) \geqslant 0$. Hence it follows that a $p^{[h]}$-dimensional vector l^0 and a number $\nu^0 \geqslant 0$, $\sigma^{[h]*}(l^0, \nu^0) \leqslant 1$ exist such that

$$\sigma^{[h]}(D^{[h]}(x_* - g^{[h]}),\ e(t_* + 0, x_*; \Delta_k)) =$$
$$= \max_{\sigma^{[h]*}(l,\nu) \leqslant 1} [\langle l, D^{[h]}(x_* - g^{[h]})\rangle + \nu e(t_* + 0, x_*; \Delta_k)] = \langle l^0, D^{[h]}(x_* - g^{[h]})\rangle + \nu^0 e(t_* + 0, x_*; \Delta_k) \tag{3.2}$$

We define a vector

$$m^0 = \nu^0 m_*^0 + X^T(t_*, \vartheta) D^{[h]T} l^0 \tag{3.3}$$

It follows from (2.6) and (3.3) that $m^0 \in G_1(t_*, \tau_1 - 0)$, and it then follows from (2.7) that $\varphi_1(t_*, \tau_1 - 0, m^0) \geqslant \nu^0 \varphi_1(t_*, \tau_1 + 0, m_*^0) - \langle l^0, D^{[h]} g^{[h]}\rangle$.

Thus, by (2.9) and (3.1)–(3.3), we obtain

$$e(t_* - 0, x_*; \Delta_k) \geqslant \alpha_1^-(t_*, x_*, m^0) \geqslant \langle l^0, D^{[h]}(x_* - g^{[h]})\rangle + \nu^0 \alpha_1^+(t_*, x_*, m_*^0) =$$
$$= \sigma^{[h]}(D^{[h]}(x_* - g^{[h]}),\ e(t_* + 0, x_*; \Delta_k)) \tag{3.4}$$

On the other hand, in the case under consideration, by the constructions (2.6) and (2.7) (with $j = 1$), for each vector $m \in G_1(t_*, \tau_1 - 0)$ at least one triple $\{\nu, m_*, l\}(m) = \{\nu(m), m_*(m), l(m)\}$ exists such that

$$\begin{aligned} &\nu(m) \geqslant 0, \quad m_*(m) \in G_1(t_*, \tau_1 + 0), \quad \sigma^{[h]*}(l(m), \nu(m)) \leqslant 1 \\ &\nu(m) m_*(m) + X^T(t_*, \vartheta) D^{[h]T} l(m) = m \\ &\varphi_1(t_*, \tau_1 - 0, m) = \nu(m) \varphi_1(t_*, \tau_1 + 0, m_*(m)) - \langle l(m), D^{[h]} g^{[h]}\rangle \end{aligned} \tag{3.5}$$

In turn, by (2.9) a vector $m_0 \in G_1(t_*, \tau_1 - 0)$ exists such that

$$e(t_* - 0, x_*; \Delta_k) = \alpha_1^-(t_*, x_*, m_0) \tag{3.6}$$

Let $\{\nu_0, m_{*0}, l_0\} = (\nu, m_*, l\}(m_0)$ be the triple of (3.5) corresponding to this vector m_0. Then

$$e(t_* + 0, x_*; \Delta_k) \geqslant \alpha_1^+(t_*, x_*, m_{*0}) \tag{3.7}$$

It now follows from (3.6), taking note of (2.9) and (3.5) (for $m = m_0$), that

$$e(t_* - 0, x_*; \Delta_k) = \nu_0 \alpha_1^+(t_*, x_*, m_{*0}) + \langle l_0, D^{[h]}(x_* - g^{[h]})\rangle \leqslant$$
$$\leqslant \langle l_0, D^{[h]}(x_* - g^{[h]})\rangle + \nu_0 e(t_* + 0, x_*; \Delta_k) \leqslant \sigma^{[h]}(D^{[h]}(x_* - g^{[h]}),\ e(t_* + 0, x_*; \Delta_k)) \tag{3.8}$$

Relationships (3.4) and (3.8) prove the statement of Lemma 1 for the case $t_* = t^{[h]}$, where $h = h(t_*)$.

Lemma 2 (u- and υ-stability). Suppose that the realized position is $\{t_*, x_*\} \in K$, $t_* < \vartheta$ and that a partition Δ_k (2.1) of the time interval $[t_*, \vartheta]$ has been chosen. Then, for any relation of the noise

$$v_*[t_*[\cdot]t^*) = \{v_*[t] = v_* \in Q,\ t_* \leqslant t < t^*\} \tag{3.9}$$

in the case of u-stability (of the control

$$u_*[t_*[\cdot]t^*) = \{u_*[t] = u_* \in P,\ t_* \leqslant t < t^*\} \tag{3.10}$$

in the case of υ-stability), where $t^* = \tau_2$ is the second point of Δ_k, an admissible realization of the control $u[t_*[\cdot]t^*) = \{u[t] \in P, t_* \leqslant t < t^*\}$ (of the noise $\upsilon[t_*[\cdot]t^*) = \{\upsilon[t] \in Q, t_* \leqslant t < t^*\}$) exists such that, from an initial position $[t_*, x_*]$, the realizations $u[t_*[\cdot]t^*)$ and $\upsilon_*[t_*[\cdot]t^*)$ ($u[t_*[\cdot]t^*)$ and $\upsilon[t_*[\cdot]t^*)$) steer system (1.1) to a position $\{t^*, x^* = x[t^*]\} \in K$ such that

$$e(t_* + 0, x_*; \Delta_k) \geqslant e(t^* - 0, x^*; \Delta^*_{k^*}) \tag{3.11}$$

in the case of u-stability (or

$$e(t_* + 0, x_*; \Delta_k) \leqslant e(t^* - 0, x^*; \Delta^*_{k^*}) \tag{3.12}$$

in the case of υ-stability).

Here $\Delta^*_{k^*} = \Delta^*_{k^*}\{\tau^*_j\}$ is the partition of the time interval $[t_*, \vartheta]$ generated by the points of Δ_k such that $\tau^*_j = \tau_{j+1}$, $\tau_{j+1} \in \Delta_k$, $j = 1, \ldots, k^* + 1$, $k^* = k - 1$.

Proof of u-stability. Let $W = W(t^*; t_*, x_*, \upsilon_*)$ be the reachable domain (over all admissible realizations $u[t_*[\cdot]t^*)$) of system (1.1) at time t^* from the position $[t_*, x_*]$, given the realization $\upsilon[t_*[\cdot]t^*)$ of (3.9). We have to show that a vector $x^* \in W$ exists satisfying inequality (3.11).

Using the Cauchy formula for solutions of Eq. (1.1) (with the substitution $u[t] \to u$ and $\upsilon_* \to \upsilon$), we obtain

$$W = \left\{ w:\ w = X(t^*, t_*)x_* + \int_{t_*}^{t^*} X(t^*, \tau) f(\tau, u[\tau], v_*) d\tau,\ \{u[\tau] \in P,\ t_* \leqslant \tau < t^*\} \right\} \tag{3.13}$$

It follows from known facts in the theory of the integration of multivalued mappings (see, e.g. [12, p. 349]) that the set W is non-empty, convex and compact in R^n.

We now note that, by virtue of our notation and the relationship between the partitions $\Delta_k\{\tau_j\}$ and $\Delta^*_k\{\tau^*_j\}$, the construction (2.4) implies the following identities

$$\begin{gathered} G_1(t^*, \tau^*_1 - 0) = G_2(t_*, \tau_2 - 0) = G_1(t_*, \tau_1 + 0) \\ \varphi_1(t^*, \tau^*_1 - 0, m) \equiv \varphi_2(t_*, \tau_2 - 0, m) \end{gathered} \tag{3.14}$$

Consider the function $\alpha_1^-(t^*, w, m)$, $w \in W$, $m \in G_1(t_*, \tau_1 + 0)$ defined on the basis of the partition Δ^*_k by the second equality of (2.9). It is bounded, concave and upper semi-continuous with respect to m for every fixed w, and convex and continuous with respect to w for every fixed m. Consequently (see, for example, [13, p. 31]), a saddle point $\{w^0 \in W, m^0 \in G_1(t_*, \tau_1 + 0)\}$ exists such that

$$\alpha_1^-(t^*, w^0, m^0) = \max_{m \in G_1(t_*, \tau_1 + 0)} \alpha_1^-(t^*, w^0, m) \tag{3.15}$$

$$\langle m^0, X(\vartheta, t^*) w^0 \rangle = \min_{w \in W} \langle m^0, X(\vartheta, t^*) w \rangle \tag{3.16}$$

We will show that the required inequality (3.11) holds for the vector $x^* = w^0 \in W$. Let $u^0[t_*[\cdot]t^*)$ be the realization of the control, which by (3.13) corresponds to the vector w^0. Then, by (3.13), taking into consideration that here by the measurable selection theorem of [14, p. 26], the minimization operation operation may be taken inside the integral sign, we conclude from (3.16) that

$$\int_{t_*}^{t^*} \langle m^0, X(\vartheta, \tau) f(\tau, u^0[\tau], v_*) \rangle d\tau = \int_{t_*}^{t^*} \min_{u \in P} \langle m^0, X(\vartheta, \tau) f(\tau, u, v_*) \rangle d\tau \tag{3.17}$$

Further, taking (3.14) and (3.15) into account, we conclude from (2.9) that

$$e(t^* - 0, w^0; \Delta^*_{k^*}) = \alpha_2^-(t_*, x_*, m^0) + \int_{t_*}^{t^*} \langle m^0, X(\vartheta, \tau) f(\tau, u^0[\tau], \upsilon_*)\rangle d\tau \tag{3.18}$$

On the other hand, since $m^0 \in G_1(t_*, \tau_1 + 0)$, we conclude from (2.9), using (2.4) (with $j = 1$) and the majorizing property of the hull $\varphi_1(t_*, \tau_1 + 0, m)$ for the function $\psi_1(t_*, m)$, that

$$e(t_* + 0, x_*; \Delta_k) \geqslant \alpha_2^-(t_*, x_*, m^0) + \Delta\psi_1(t_*, m^0) \tag{3.19}$$

Now, taking (2.2) into consideration, we conclude that the difference between the left-hand sides of (3.18) and (3.19) does not exceed the difference between the left- and right-hand sides of (3.17). This proves inequality (3.11) for $x^* = w^0 \in W$, thus establishing u-stability.

Proof of υ-stability. It follows from (2.9), via Carathéodory's theorem [15, p. 155] for the upper convex hull $\varphi_1(t_*, \tau_1 + 0, m)$ of the function $\psi_1(t_*, m)$ (see (2.4) with $j = 1$) that a vector $m^0 \in G_1(t_*, \tau_1 + 0)$ exists such that

$$e(t_* + 0, x_*; \Delta_k) = \alpha_1^+(t_*, x_*, m_0) = \alpha_2^-(t_*, x_*, m_0) + \Delta\psi_1(t_*, m_0) \tag{3.20}$$

We choose a sample of the noise $\upsilon[t_*[\cdot]t^*)$ to satisfy the condition

$$\langle m_0, X(\vartheta, t) f(t, u_*, \upsilon[t])\rangle = \max_{\upsilon \in Q} \langle m_0, X(\vartheta, t) f(t, u_*, \upsilon)\rangle \tag{3.21}$$

$$t_* \leqslant t < t^*$$

By the measurable selection theorem [14, p. 26], such an admissible sample $\upsilon[t_*[\cdot]t^*)$ exists for any fixed $u_* \in P$.

The samples $u[t_*[\cdot]t^*)$ (3.10) and $\upsilon[t_*[\cdot]t^*)$ (3.21) steer system (1.1) from the position $[t_*, x_*]$ to a position $\{t^*, x^* = x[t^*]\} \in K$, for which, by (2.9), using identities (3.14) and the Cauchy formula, we obtain

$$e(t^* - 0, x^*; \Delta^*_{k^*}) \geqslant \alpha_1^-(t^*, x^*, m_0) = \alpha_2^-(t_*, x_*, m_0) + \int_{t_*}^{t^*} \langle m_0, X(\vartheta, t) f(t, u_*, \upsilon[t])\rangle dt \tag{3.22}$$

Inequality (3.12) now follows from (3.20) and (3.22), after taking into account (1.2), (2.2) and (3.21). This completes the proof of Lemma 2.

Now, in order to verify the truth of the theorem, it will suffice, choosing a number $\varepsilon > 0$ and a partition $\Delta_k\{\tau_j\}$ (2.1) as required, to consider the evolution of the quantities $e(\tau_j - 0, x[\tau_j]; \Delta^{(j)}_{k(j)})$ $(j = 1, \ldots, k + 1)$, where $\Delta^{(j)}_{k(j)} = \Delta^{(j)}_{k(j)}\{\tau_i^{(j)}\} = \tau_i^{(j)} = \tau_{i+j-1} \in \Delta_k, i = 1, \ldots, k^{(j)} + 1, k^{(j)} = k - j + 1\}$, on the one hand, along the motion of system (1.1) that is realized when Player II is guided by the noise-formation law $V^\circ = \{\upsilon^\circ(\cdot), \varepsilon, \Delta_k\{\tau_j\}\}$ (see (1.7)) while Player I, relying on information about the actually realized position $\{\tau_j, x[\tau_j]\}$ and the designated noise sample $\upsilon^\circ[t] = \upsilon^\circ(\tau_j, x[\tau_j], \varepsilon)$, $\tau_j \leqslant t < \tau_{j+1}$, chooses a control $u[\tau_j[\cdot]\tau_{j+1}$ at each step in accordance with Lemma 2 (u-stability); on the other hand, the evolution must also be considered along the motion realized when Player I has chosen the control law $U^\circ = \{u^\circ(\cdot), \varepsilon, \Delta_k\{\tau_j\}\}$ (see 1.6)) and Player II designates the noise $\upsilon[\tau_j[\cdot]\tau_{j+1})$ in accordance with Lemma 2′ (υ-stability). Throughout, note should be taken of the equalities established in Lemma 1, the fact that $\sigma^{[i]}(y^{[i]}, \beta)$ $(i = 1, \ldots, N - 1)$ are non-decreasing functions of β, and relationships (1.5).

I wish to thank N. N. Krasovskii for his advice and comments.
This research was supported financially by the Russian Foundation for Basic Research (97-01-00160).

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Translated by D.L.